(4) Runaways → Strong $E$ case

→ message is that stationary state not always possible for strong perturbation

→ basic idea

– consider electron at $v \leq v_{Th}$

– in one mfp, increment in speed is

$$\delta v \sim \frac{qE}{m_e} \tau_{mfp} \sim \frac{qE}{m_e} \frac{l_{mfp}}{v_{Th}}$$

$$\sim \frac{qE}{m_e}\left(\frac{1}{n\sigma v_{Th}}\right)$$

but $\boxed{\sigma \sim 4\pi (e^2)^2 \ln\Lambda / m_e^2 v_{Th}^4}$

$(\sim \pi b_0^2 \ln\Lambda)$

so

$$\delta v \sim \frac{v_{The}^3 \, m E}{4\pi e^3 \ln\Lambda \, n_0}$$

$$\frac{\delta V}{V_{Th}} \sim V_{Th}^2 m_e E \Big/ 4\pi e^3 \ln\Lambda\, n_0$$

$\delta V / V_{Th} \sim 1 \Rightarrow V_{crit}$ defined.
$\Rightarrow E_{crit}$ "

$$\left\{ \begin{array}{l} V_{crit} \sim \left( \dfrac{n e^3 \ln\Lambda}{m^2 E} \right)^{1/2} \\[2ex] \boxed{E_{crit} \sim \dfrac{4\pi e^3 \ln\Lambda\, n_0}{T_e}} \end{array} \right.$$

$\rightarrow$ critical electric field (Dreicer Field) for $\delta V \sim V_{Th}$

Now, $\delta V \sim \left( \dfrac{V_{Th}}{\delta V} \right)^2 V_{Th}^2$

and for $\delta V > V_{Th}$, $\delta V$ replaces $V_{Th}$ in cross-section, etc.

so $\rightarrow$ momentum increment:

$$\Delta p \sim eE \frac{l_{mfp}}{V_{Th}} \rightarrow \frac{eE}{\delta V\, n\sigma_e(\delta V)}$$

$\Rightarrow$ $\underline{E} > \underline{E}_{crit}$

$$\Delta p \sim \frac{eE}{\delta v\, n\, \nu_e(\delta v)}$$

$$\sim \frac{eE \quad m^2 (\delta v)^4}{\delta v\, n\, \ln\Lambda\, (e^2)^2\, n_0}$$

$$\sim \frac{eE\, m^2\, \delta v^3}{n \ln\Lambda\, e^3\, n_0}$$

$$\sim m\, \delta v \left(\frac{\delta v}{v_{crit}}\right)^2$$

$\Rightarrow$ $\delta v \sim v_{crit}$, $\Delta p \sim m\, \delta v$

$\Rightarrow$ electrons accelerated without limit, if speed, $E$ high enough

$\Rightarrow$ $E > E_{crit} = 4\pi e^3 \ln\Lambda\, n_0 / T_e$

$\sim$ bulk "runs away"

$\Rightarrow$ Dreicer field for runaway.

→ $E < E_{crit}$ ⇒ tail runs away.

→ Time scales

$$\frac{\tau_{i,i}}{\tau_{e,i}\ {}_{e,e}} \sim \sqrt{m_i/m_e}$$

$$\frac{d}{dt} T_e = -\frac{1}{\tau^E_{e,i}}(T_e - T_i)$$

$$\tau^E_{e,i} \sim \frac{m_e}{m_i}$$ ⇒ equilibration time longest.

$$\tau_{ee,\,ei} < \tau_{i,i} < \tau^E_{e,i}$$

(b) Applications – Dynamic Screening – { Collective Enhancement of Collisional Relaxation

Consider form $B_{\alpha\beta}$:

$$B_{\alpha\beta} = 2(q_e q'_e)^2 \int_{-\infty}^{+\infty} \int_{k<k_{max}} \delta(\omega - \underline{k}\cdot\underline{v})\delta(\omega - \underline{k}\cdot\underline{v}') \frac{k_\alpha k_\beta \, d^3k \, d\omega}{k^4 |\epsilon(\underline{k},\omega)|^2}$$

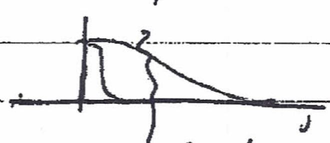
i.e. $\underline{k}\cdot\underline{v} = \underline{k}\cdot\underline{v}' \Rightarrow \underline{k}\cdot(\underline{v}-\underline{v}') = 0$

Consider stable, 2-species plasma. Then have two collective resonances (i.e. weakly damped waves):
(no shift)

① → electron plasma waves; $\omega/k > v_{Te}$

② → ion acoustic waves; $v_{Ti} < \frac{\omega}{k} < v_{Te}$
(no shift fe)

① → tail of $\langle f \rangle_e$ ⇒ relatively few particles, little role in collision dynamics

② → if $T_e \gg T_i$  not 

weak damping
$\langle f \rangle$ small

heavy ion, electron damping

∴ for $T_e \gg T_i$, ion acoustic resonance may enhance collisional relaxation (weakly damped modes)

To show, exploit 'pole approximation': (collective resonance enhancement of $B$)

$$\frac{1}{|\epsilon|^2} = \frac{1}{|\epsilon_r|^2 + |\epsilon_{IM}|^2}$$

$$\cong \frac{1}{\left[\left|(\omega-\omega_k)\left.\frac{\partial\epsilon}{\partial\omega}\right|_{\omega_k}\right|^2 + |\epsilon_{IM}|^2\right]}$$ (damping → resonance linewidth)

$$\cong \underbrace{\frac{\pi}{|\epsilon_{IM}|}\,\delta(\epsilon_r)}_{\text{wave resonance}}$$

i.e. $\begin{cases} \epsilon_r = 0 \rightarrow \text{resonance location} \\ \epsilon_{IM} \rightarrow \text{resonance size/width} \end{cases}$

and note for electron - electron collisions:

$\omega \ll \underline{k}\cdot\underline{v}$, $\underline{k}\cdot\underline{v}'$ ; due $\omega < \underline{k}\cdot v_{Te}$ ordering

⇒ so

$$B_{\alpha\beta} \cong 2\pi q^4 \int_{-\infty}^{+\infty}\int \delta(\underline{k}\cdot\underline{v})\,\delta(\underline{k}\cdot\underline{v}')\,\delta(\epsilon_r)\,\frac{k_\alpha k_\beta}{|\epsilon_{IM}|}\,d^3k\,d\omega$$

Change variables:

$\kappa = \underline{k}\cdot\hat{n}$ (kappa) $\hat{n}$ unit along $\underline{v}\times\underline{v}'$

$k_1 = \underline{k}\cdot\underline{v}$

$k_2 = \underline{k}\cdot\underline{v}'$

then: $d^3k = d\kappa\, dk_1\, dk_2 / |\underline{v}\times\underline{v}'|$

so

$\Rightarrow$

$$B_{\alpha\beta} = \frac{2\pi e^4 n_\alpha n_\beta}{|\underline{v}\times\underline{v}'|}\, 2\int_{k>0} dk \int_{-\infty}^{+\infty} d\omega \,\frac{\delta(\epsilon_r(k,\omega))}{k^2|\epsilon_{IM}|}$$

i.e. collapse $k_1, k_2$ integrals

Now, $\epsilon_r = 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{1}{k^2\lambda_D^2}$

$\omega = kc_s/(1+k^2\lambda_{De}^2)^{1/2}$

ion-acoustic wave

electron L.D. $\omega \ll kv_{Te}$ I.L.D

$$\rightarrow \epsilon_{IM} = \sqrt{\frac{\pi}{2}}\,\frac{\omega}{k^3}\left(\frac{1}{\lambda_{De}^2 v_{Te}} + \frac{1}{\lambda_{Di}^2 v_{Ti}}\, e^{-\omega^2/2k^2v_{Ti}^2}\right)$$

dominant contribution from shortwavelength $(k\lambda_{De} > 1)$

$\rightarrow$ (i.e. max $1/\epsilon_{IM}$)

$\therefore$

$$\begin{cases} \omega \simeq \omega_{pi} \\ \epsilon_{IM} = \sqrt{\frac{\pi}{2}}\,\frac{\omega_{pi}}{k^3}\left(\frac{1}{\lambda_{De}^2 v_{Te}} + \frac{1}{\lambda_{Di}^2 v_{Ti}}\, e^{-1/k^2\lambda_{Di}^2}\right) \end{cases}$$

$\Rightarrow$

$$\delta(\epsilon_r) = \delta\left(1 - \omega_{pi}^2/\omega^2\right) \simeq \frac{1}{2}\,\omega_{pi}\left[\delta(\omega-\omega_{pi}) + \delta(\omega+\omega_{pi})\right]$$

$$\Rightarrow \boxed{B_{\alpha\beta} = \frac{4\pi e^4 \omega_{pi}\, n_\alpha n_\beta}{|\underline{v}\times\underline{v}'|} \int \frac{dK}{K^2 |\epsilon_{T_i}(\omega_{pi}, K)|}}$$

$$\varepsilon = K^2 \lambda_{Di}^2$$

$$\epsilon_{I\Pi} = \sqrt{\frac{\pi}{2}}\, \frac{\omega}{k^3} \left\{ \frac{1}{\lambda_{De}^2 v_{Te}} + \frac{1}{\lambda_{Di}^2 v_{Ti}}\, e^{-\omega_{pi}^2/2k^2 v_{Ti}^2} \right\}$$

$$\therefore \left\{ \begin{aligned} B_{\alpha\beta} &= n_\alpha n_\beta \frac{2\sqrt{2\pi}\, e^4 v_{Te} \lambda_{De}^2}{|\underline{v}\times\underline{v}'|\ \lambda_{Di}^2} \int d\varepsilon \Big/ \left[1 + \exp\left(-\frac{1}{2\varepsilon} + \frac{L_1}{2}\right)\right] \\ L_1 &= \ln\left(\left(T_e/T_i\right)\left(v_{Te}^2/v_{Ti}^2\right)\right) \end{aligned} \right.$$

Now:

(i) $v_{Ti} < \frac{\omega}{k} < v_{Te} \Rightarrow \overset{\left(\omega_{pi}\lambda_{Di}/\omega_{pe}\lambda_{De}\right)^2}{\cancel{1}} \ll \varepsilon < 1$

(ii) $L_1 \gg 1 \Rightarrow$ expand $O(1/L_1)$

i.e. dominant contribution when:

$$\exp\left(-1/2\varepsilon + \frac{L_1}{2}\right) \ll 1 \Rightarrow \varepsilon \lesssim 1/L_1$$

note: $\left[1 + \exp(-1/2\varepsilon + L_1/2)\right]^{-1} \equiv$ i.e. denominator recall:

$$\boxed{B_{\alpha\beta} = n_\alpha n_\beta \left[ \frac{2\sqrt{2\pi}\, e^4 v_{Te}\, \lambda_{De}^2}{|\underline{v}\times\underline{v}'|\; \lambda_{Di}^2} \right] (1/L_1)}$$

Now;

above ($v \ll v_{Te}$) ($v \sim v_{Te}$)

$$B_{\alpha\beta} = B_{\alpha\beta}^{\text{collective}} + B_{\alpha\beta}^{\text{coulomb}}$$

(collective resonance dominant) (ballistic spectrum dominant)

as peaks in spectrum not coincident.

$$B_{\alpha\beta}^{\text{coulomb}} = \frac{2\pi e^4}{|\underline{v}-\underline{v}'|} L \left[ \delta_{\alpha\beta} - \frac{(v_\alpha - v_\alpha')(v_\beta - v_\beta')}{|\underline{v}-\underline{v}'|^2} \right]$$

$$\cong \frac{2\pi e^4}{v_{Te}} L_{\text{coulomb}}$$

$B^{\text{collective}} \geq B^{\text{coulomb}}$ if:

$$\boxed{\frac{T_e}{T_i L_1} \geq L_{\text{coulomb}}}$$

Coulomb log.

Criterion for dominance of collective effect enhanced scattering (need $T_e \gg T_i$)